

MAS 117

Introduction to Algebra

Solutions 2

January 2007

- 1 (a) For $n = 1$, the left-hand expression is $1/3$, and so is the right-hand expression. Suppose that the equation holds for the value n ; that is, suppose that

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{(2n-1) \times (2n+1)} = \frac{n}{2n+1}.$$

Then

$$\begin{aligned} \frac{1}{1 \times 3} + \cdots + \frac{1}{(2n+1) \times (2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{n(2n+3) + 1}{(2n+1)(2n+3)} \\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}, \end{aligned}$$

[On the left, we have the sum of $n+1$ terms, of which the first n form the sum whose value we know by assumption. So the total is this sum plus the $n+1$ st term.] The final expression is what is obtained by substituting $n+1$ for n in the expression on the right. So we have done the inductive step.

- (b) For $n = 4$, the two sides of the inequality are both equal to 256, so it is true. [Note that it is not true for $n = 1, 2, 3$.]

For the inductive step, we see that in going from n to $n+1$, the left-hand expression is multiplied by 4. If we can show that the right-hand expression is multiplied by a factor which is not more than 4, we will get the required result. So we have to prove that $(n+1)^2/n^2 \leq 4$. The proof goes like this: If $n \geq 4$, then

$$(n+1)^2 = n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 = 4n^2,$$

since $2n \leq 2n^2$ and $1 \leq n^2$.

So, assuming the result for n , we have $4^n \geq 16n^2$, and so

$$\begin{aligned} 4^{n+1} &= 4 \cdot 4^n \\ &\geq 4 \cdot 16n^2 \text{ by the induction hypothesis} \\ &= 16 \cdot 4n^2 \\ &\geq 16(n+1)^2 \text{ by what we just proved,} \end{aligned}$$

so the inductive step is done.

2 (a) It is a definition (of the concept of scalar product of two vectors).

(b) Many answers are possible. Here is my suggestion. The best strategy is just to describe:

- The scalar product of two vectors is a scalar defined in terms of their geometry.

Another approach would be to paraphrase (but this tends to get long-winded):

- The scalar product of two vectors is the product of their lengths and the cosine of the angle between them (unless one of the vectors is zero, in which case it is zero).

(c) The bold **0** is the zero vector, whereas the ordinary 0 is the zero real number.

3

$$\begin{aligned} (a+bi)(c+di) &= (ac-bd) + (ad+bc)i, \\ (c+di)(a+bi) &= (ca-db) + (cb+da)i, \end{aligned}$$

and these two expressions are equal (by the commutative laws for real numbers).

$$\begin{aligned} (a+bi)((c+di)(e+fi)) &= (a+bi)((ce-df) + (cf+de)i) \\ &= (a(ce-df) - b(cf+de)) + (a(cf+de) + b(ce-df))i, \\ ((a+bi)(c+di))(e+fi) &= ((ac-bd) + (ad+bc)i)(e+fi) \\ &= ((ac-bd)e - (ad+bc)f) + ((ac-bd)f + (ad+bc)e)i, \end{aligned}$$

and these are equal after some rearrangement (using the commutative, associative and distributive laws).

4 Suppose that $x^4 + 4 = (x^2 + ax + b)(x^2 + cx + d)$. Multiplying out and equating coefficients, we have

$$a+c=0, \quad b+ac+d=0, \quad ad+bc=0, \quad bd=4.$$

The first equation is $c = -a$, and the second equation then shows that $a(d - b) = 0$, so that either $a = 0$ or $b = d$.

If $a = 0$, then $b + d = 0$ and $bd = 4$, so $b^2 = -4$, impossible. So $b = d = \pm 2$.

If $b = d = -2$ then $ac = 4$, so $a^2 = -4$, which is impossible. So $b = d = 2$, and $a^2 = 4$, $a = \pm 2$.

So finally $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$.

This can be verified by multiplying out the polynomials.

Note: I was very pleased that several people found the following nice proof.

$$\begin{aligned}x^4 + 4 &= (x^4 + 4x^2 + 4) - (4x^2) \\&= (x^2 + 2)^2 - (2x)^2 \\&= (x^2 + 2x + 2)(x^2 - 2x + 2).\end{aligned}$$